TANGENT LIFTING OF DEFORMATIONS IN MIXED CHARACTERISTIC

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Abstract. This article presents a new approach to the unobstructedness result for deformations of Calabi-Yau varieties by introducing the *tangent lifting property* for a functor on artinian local algebras. The verification that the deformation functor of a Calabi-Yau variety in characteristic 0 fulfills the tangent lifting property uses, just as the verification of the T^1-lifting property, the degeneration of the Hodge to de Rham spectral sequence. Our main use of our methods is however to the mixed characteristic case. In that case to be able to verify the conditions needed one needs an extension of the criterion introducing divided powers.

Fix a complete noetherian local ring Λ with residue field $\mathbf{k}$, and denote by C_Λ, respectively $\widehat{C}_\Lambda$, the category of Artin, respectively complete, local Λ-algebras with residue field $\mathbf{k}$. Suppose that F is a semi-homogeneous cofibered groupoid (cf. [SGA7I]) over C_Λ such that $F(\mathbf{k})$ is equivalent to a point so that $\overline{F} := \pi_0 F$, the functor of isomorphism classes of objects of F, has a hull R, say. Define $A_n = \mathbf{k}[t]/(t^{n+1})$ and $B_n = A_n[\epsilon]/(\epsilon^2)$. Then F is said to have the T^1*-lifting property* if the natural map $F(B_{n+1}) \to F(B_n) \times_{F(A_n)} F(A_{n+1})$, where the fibre product is the 2-fibre product, is essentially surjective. Ran ([R92]), Kawamata ([K92]) and Fantechi and Manetti ([FM99]) have shown that if $\Lambda = \mathbf{k}$ and $\operatorname{char}\mathbf{k} = 0$, then R is formally smooth over $\mathbf{k}$ if F has the T^1-lifting property. (The T^1-lifting property can also be phrased as follows. Given $X \in F(A)$, define $T^1(X/A)$ as the set of isomorphism classes of pairs (Y, ψ), where $Y \in F(A[\epsilon])$ and ψ is an isomorphism $\psi\colon Y_{|A} \to X$. Then the T^1-lifting property is equivalent to the surjectivity of $T^1(X_n/A_n) \to T^1(X_{n-1}/A_{n-1})$.) Moreover, the T^1-lifting property is satisfied by, e.g., deformations of Calabi-Yau varieties.

In this article we propose an approach to the question of whether R is formally smooth that puts different conditions on the functor. Even though this approach gives a new proof of the unobstructedness for deformations of Calabi-Yau varieties we shall mainly be interested in the case when Λ has mixed characteristic or has positive characteristic. Roughly speaking, in the mixed case this involves proving smoothness by (1) lifting the whole tangent space at once, rather than extending a given tangent direction, and (2) assuming the existence of some lifting to char. zero. For deformations of Calabi-Yau varieties (defined as smooth varieties with trivial dualizing sheaf) in characteristic zero the fact that these conditions are fulfilled follows in the same way as the T^1-lifting property but make the verification of smoothness simpler. They are inspired by the fact that in characteristic zero lifting of tangent vectors to vector fields implies smoothness, a fact that is central in, for instance, Cartier's proof that a group scheme in characteristic zero is smooth. Our approach has the added advantage of not requiring that $\overline{F}$ comes from a cofibered groupoid (in practice this seems almost always to be the case so it is not clear how great an advantage that is) and hence unless explicitly stated we shall assume only that $\overline{F}$ is a functor from C_Λ to the category of sets. We shall say that $\overline{F}$ has *the tangent lifting property* (abbreviated to TLP) if for all $A \in C_\Lambda$ the natural map $\overline{F}(A[\epsilon]) \to \overline{F}(A) \times \overline{F}(\mathbf{k}[\epsilon])$ is surjective.

Remark: (1) It is not clear to us what the precise logical relations between the notions of T^1-lifting and tangent lifting are, although in characteristic zero T^1-lifting implies tangent lifting *a posteriori* as tangent lifting is true when the hull

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is smooth. It should also be noted that the reason that we do not need to assume that $\overline{F}$ has an underlying groupoid for the tangent lifting property is that contrary to the case of T^1-lifting only the product $\overline{F}(A) \times \overline{F}(\mathbf{k}[\epsilon])$ and no fibre product is involved.

(2) Schröer [Sc01] has results similar to the ones presented here, but with stronger assumptions. Just as our approach gives an alternative in the characteristic zero to previous ones our method gives a new approach independent of Schröer's in the mixed characteristic case. See the remark after Theorem C below for a more thorough explanation.

A first re-formulation of this condition is given in the following lemma.

Lemma 0.1 *When $\overline{F}$ is $\pi_0 F$ of a cofibered groupoid the tangent lifting property is equivalent to the surjectivity of $T^1(X/A) \to T^1(X_0/\mathbf{k})$ for all $X \in F(A)$.*

PROOF: Consider the restriction map $\pi : F(A[\epsilon]) \to F(A)$. Then for any $X \in F(A)$, there is a forgetful map $T^1(X/A) \to \pi^{-1}(X)$ and this map is surjective. Since $T^1(X_0/\mathbf{k})$ is identified with $F(\mathbf{k}[\epsilon])$, the lemma is now (even more) obvious. □

If X_0 is a smooth variety either in char. zero or in char. p and F is the deformation groupoid we may identify $T^1(X/A)$ with $H^1(X, T_{X/A})$, which is isomorphic to $H^1(X, \Omega^{n-1}_{X/A})$ provided that $\omega_{X/A}$ is trivial. Thus the tangent lifting condition may be analysed through Hodge cohomology. In particular it is always fulfilled for Calabi-Yau varieties (that is, those for which ω is trivial) in characteristic zero. More generally, the deformation functor of a complete smooth variety X_0 has the tangent lifting property if and only if for every deformation $f\colon X \to \mathbf{Spec}\, A$ the specialization homomorphism $R^1 f_* T_{X/A} \to H^1(X_0, T_{X_0})$ is surjective.

Notation: We shall abuse language by failing to distinguish between smoothness and formal smoothness, derivations and continuous derivations, and differentials and continuous differentials. We shall furthermore, for a sheaf of rings $\mathcal{O}$ let $\mathcal{O}[\epsilon]$ denote $\mathcal{O}[x]/(x^2)$, where ϵ is the residue of x. Similarly if X is a scheme we put $X[\epsilon] := \mathbf{Spec}\, \mathcal{O}[\epsilon]$.

To avoid confusion in the case where the involved schemes are non-reduced let us make explicit that we say that a map $f\colon X \to S$ of (formal) schemes is *dominant* if the induced map $f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ is injective.

We shall show that the tangent lifting property for $\overline{F}$ is equivalent to a tangent lifting property for a deformation hull, namely that any tangent vector (at the closed point) lifts to a vector field. In characteristic zero this immediately implies that R is formally smooth whereas in positive characteristic it implies that defining equations for R can be assumed to be p'th powers. Just as in [Sc01], in the case of mixed characteristic we shall need to have at least one lifting to characteristic zero. Since this involves rings in $\widehat{C}_\Lambda$ rather than C_Λ, we make a convention: Given a functor $\overline{F} : C_\Lambda \to \underline{\mathrm{Sets}}$ and $R \in \widehat{C}_\Lambda$, we write $R = \varprojlim A_\alpha$, with $A_\alpha \in C_\Lambda$, and then define $\overline{F}(R)$ to be the set of pro-objects $\{\xi_\alpha \in \overline{F}(A_\alpha)\}$.

Theorem A *Assume that $\overline{F}$ has the tangent lifting property and let R be a hull for it.*

(1) If Λ contains $\mathbb{Q}$, then R is smooth.

(2) If $\Lambda = \mathbf{k}$ a field and $\operatorname{char} \mathbf{k} = p > 0$, then $R \cong \mathbf{k}[[x_1, \dots, x_n]]/(f_1, \dots, f_r)$ for some f_i in the maximal ideal of $\mathbf{k}[[x_1, \dots, x_n]]$ that are of the form $f_i = g_i(x_1^p, \dots, x_n^p)$. In particular, if $\mathbf{k}$ is perfect and $\mathbf{Spec}\, R$ is generically reduced, then it is smooth.

(3) If Λ is torsion free and $f\colon \Lambda \to \Lambda_1$ is an injective local map in $\widehat{C}_\Lambda$ such that $\overline{F}(\Lambda_1)$ is non-empty, then R is formally smooth over Λ.

To give the reader some feeling for this result let us quickly sketch the proof in characteristic zero: The tangent lifting means that every horizontal tangent vector lifts to a horizontal vector field, i.e., to a Λ-derivation. By a result of Zariski, Lipman and Nagata this implies that R is smooth.

Actually the result of these authors is more general in that they give a structure result when only some tangent vectors lift. An analogous result would be potentially useful also in mixed and positive characteristic and we do indeed present such results (and we will in fact make use of the positive characteristic version in a forthcoming article).

When trying to apply this theorem to deformations of Calabi-Yau varieties in mixed or positive characteristic we are faced with the problem that the characteristic zero proof uses the Gauss-Manin connection on de Rham cohomology to show that the de Rham cohomology is constant over infinitesimal thickenings. However, this constancy is in general true only for divided power infinitesimal thickenings. To overcome this we shall introduce the notion of the divided power tangent lifting property that requires the lifting property only for those A that have a divided power structure on their maximal ideals (compatible with a given such structure on $\mathfrak{m}_\Lambda$). In the mixed characteristic situation we again need at least one lifting over some local Λ-algebra Λ_1 that is *slightly ramified*. This is defined in the next section; for the moment, we just remark that a complete DVR of mixed characteristic p is slightly ramified over $\mathbb{Z}_p$ if and only if the absolute ramification index e satisfies $e < p$.

For the positive characteristic case, we define for an ideal I in a ring, $I^{(p)}$ to be the ideal generated by all p'th powers of I. We then say that a map of local rings $(\Lambda, \mathfrak{m}_\Lambda) \to (R, \mathfrak{m}_R)$ with $\mathfrak{m}_\Lambda^{(p)} = 0$ in characteristic p is *height* 1 *smooth* if for some n $R/\mathfrak{m}^{(p)} \cong \Lambda[t_1, \dots, t_n]/(t_1, \dots, t_n)^{(p)}$.

Theorem B *Assume that $\overline{F}$ (with hull R) fulfills the divided power tangent lifting property.*

(1) If $p\Lambda = 0$ and $\mathfrak{m}_\Lambda^{(p)} = 0$ then $\Lambda \to R$ is height 1 smooth.

(2) If Λ is torsion free and $f\colon \Lambda \to \Lambda_1$ is a slightly ramified map in $\widehat{C}_\Lambda$ such that $\overline{F}(\Lambda_1)$ is non-empty, then R is smooth over Λ.

Remark: The arguments used in the proof of this force us to permit Λ to be a pro-artinian local ring with residue field $\mathbf{k}$ rather than a complete noetherian local ring. The details of these are discussed by Gabriel, who calls them *pseudo-compact* rings, in [Ga70].

The theory of crystalline cohomology allows us to imitate the characteristic zero proof of the tangent lifting property to obtain the following result where we for simplicity have restricted Λ to be a ring of Witt vectors. (To repeat, in the case of characteristic zero we merely recover a known result).

Theorem C *Let X be a smooth and proper purely n-dimensional variety over a perfect field $\mathbf{k}$ with ω_X trivial. Let R be a hull for the deformation functor of X either over $\mathbf{k}$ or, when $\operatorname{char}\mathbf{k} > 0$ also over $\mathbf{W} = \mathbf{W}(\mathbf{k})$.*

(1) If $\operatorname{char}\mathbf{k} = 0$ then R is formally smooth.

(2) If $\operatorname{char}\mathbf{k} > 0$, $\dim_{\mathbf{k}} H^n_{DR}(X/\mathbf{k}) = \sum_{i+j=n} h^{ij}(X)$, and R is the hull for deformations over $\mathbf{k}$, then R is height 1 smooth.

(3) If $\operatorname{char}\mathbf{k} > 0$, $b_n(X) = H^n_{DR}(X/\mathbf{k}) = \sum_{i+j=n} h^{ij}(X)$ and there exists a lifting of X to a slightly ramified $\mathbf{W}$-algebra then R is a formally smooth $\mathbf{W}$-algebra.

Remark: Schröer has proved a result that is related to the third part [Sc01]. The difference is that he requires the lifting to be done already over $\mathbf{W}$ and that he does not specify any concrete conditions that makes his condition on the flatness of crystalline cohomology to be fulfilled (which we have done through the condition

on Hodge and Betti numbers). Also, it is interesting to notice that his approach is based on the T^1-lifting criterion rather than tangent lifting, as here.

This result may be combined with our results on the tangent lifting property as well as some global arguments to prove the following result.

Theorem D *Let* $\mathbf{k}$ *be a perfect field of characteristic* $p > 0$.

(1) Let $f\colon X \to S$ *be an everywhere versal family of smooth and proper purely* n*-dimensional varieties over a finite type* $\mathbf{k}$*-scheme* S *for which* ω_{X_s} *is trivial and* $\dim_{\mathbf{k}(s)} H^n_{DR}(X_s/\mathbf{k}(s)) = \sum_{i+j=n} h^{ij}(X_s)$ *for all* $s \in S$*. Then the smooth locus of* S *is open and closed.*

(2) Let $f\colon X \to S$ *be an everywhere versal family of smooth and proper purely* n*-dimensional varieties with* S *a finite type* $\mathbf{W}(\mathbf{k})$*-scheme for which* ω_{X_s} *is trivial and* $b_n(X_s/\mathbf{k}(s)) = \sum_{i+j=n} h^{ij}(X_s)$ *for all* $s \in \overline{S}$, $\overline{S} := S\bigotimes\mathbb{Z}/p$*. Then the intersection of the smooth locus of* S *with* $\overline{S}$ *equals the intersection of the closure of* $S\bigotimes\mathbf{W}[1/p]$ *with the smooth locus of* $\overline{S}$*. In particular the smooth locus of* S *is open and closed.*

For the reader's convenience let us point out that Theorems A and B are proved after Proposition 3.1, Theorem C after Proposition 4.2 and Theorem D after Proposition 4.3.

1 Divided power envelopes and some completions

In this section rings will not necessarily be noetherian. In particular, "local ring" will merely mean a ring with a unique maximal ideal.

Recall (a complete reference for this is well as other standard facts on divided powers is [Be74]) that a pair (R, I) of a commutative ring and an ideal in it is a *divided power pair* if there are maps $\gamma_n\colon I \to R$ fulfilling the relations expected of maps imitating the operations $x \mapsto x^n/n!$. For any map of pairs $(S, J) \to (R, I)$ such that (S, J) is a divided power pair we can define its *divided power hull* $\Gamma(R, I)$ which is universal for maps of the pair into a divided power pair whose composite with $(S, J) \to (R, I)$ is a divided power pair. Here, (S, J) is understood; for example, (S, J) might be $(\mathbb{Z}, 0)$. Note that we always have $n!x^n = \gamma_n(x)$ so that the divided power hull maps to the subring of $R\bigotimes\mathbb{Q}$ generated by R and the elements $x^n/n!$ for all $x \in I$. If R is the polynomial ring $\Lambda[x_1, \ldots, x_n]$ for some torsion free ring Λ and $I = (x_1, \ldots, x_n)$ then this map is an isomorphism. By functoriality this gives a description of the divided power hull for any pair (R, I). If J is a subset contained in I we define the n'th divided power ideal of J, $J^{[n]}$, as the ideal generated by the elements $\gamma_{n_1}(x_1)\ldots\gamma_{n_k}(x_k)$ for $n_1 + \ldots + n_k \geq n$. This is the smallest ideal containing J and stable under the γ_k. (Note that even if J is an ideal $J^{[1]}$ is in general distinct from J.)

There are various topological constructions that we can now make.

Definition-Lemma 1.1 *(1) Suppose that* (R, I) *is a DP pair where* R *is a topological ring,* I *is a closed ideal and the* γ_k *are continuous. If* $J \subseteq I$ *is a subset the* J-DP-adic completion *of* (R, I) *is the completion* $\hat{R}$ *in the topology defined by closure of the divided power ideals* $J^{[k]}$*. If* $\hat{I}$ *is the image of* I *in this completion then* $(\hat{R}, \hat{I})$ *is (continuous) divided power pair.*

(2) If $(R, \mathfrak{m}_R)$ *is a local ring with a divided power structure its* pro-artinian completion *is the completion in the topology defined by the collection of divided power ideals of finite colength contained in the maximal ideal. (Note that to us a local ring will not necessarily be noetherian.)*

(3) If $(R, \mathfrak{m}_R)$ *is a local ring then its* pro-artinian DP-completion *is the pro-artinian completion of its divided power hull* $\Gamma(R, \mathfrak{m}_R)$*. It will be denoted* $\hat{\Gamma}(R, \mathfrak{m}_R)$*. This is canonically isomorphic to the inverse limit of the inverse system consisting*

of finite artin R-algebras with a DP structure on their maximal ideals, where the homomorphisms in the system are DP-homomorphisms. It is also canonically isomorphic to the pro-artinian DP-completion of the $\mathfrak{m}_R$-adic completion of R.

(4) If $(R, \mathfrak{m}_R)$ is a local ring and I an ideal in R, then its I-DP-nilpotent pro-artinian DP-completion $\hat{\Gamma}_I(R, \mathfrak{m}_R)$ *is the quotient of* $\hat{\Gamma}(R, \mathfrak{m}_R)$ *by the intersection of all the ideals $I^{[n]}$. This is canonically isomorphic to the inverse limit of the inverse system consisting of finite artin R-algebras with a DP structure on their maximal ideals in which the image of I is DP-nilpotent, where again the homomorphisms in the system are DP-homomorphisms. This ring is unchanged if $(R, \mathfrak{m}_R)$ is replaced by its $\mathfrak{m}_R$-adic completion.*

Remark: (1) Even if we start with a noetherian ring, its DP envelopes and their various completions defined above are not usually noetherian. This might cast doubt on some of our arguments. However, these completions are pro-artinian, and the modules over them that we shall consider will also be pro-artinian. This will allow reduction to the artinian case. For the details of these facts, especially Nakayama's lemma, we refer to [Ga70].

(2) The pro-artinian DP-completion is not always functorial but it will be sufficiently often, as shown by the next lemma.

Lemma 1.2 *A local map of local rings $f : R \to S$ for which the residue field extension is finite extends to a continuous map between pro-artinian DP-completions. If I, J are ideals of R, S with $f(I) \subset J$, then the same holds for the nilpotent DP-completions.*

PROOF: If $S \to A$ is a ring homomorphism where A has finite length as S-module whose maximal ideal has a divided power structure then it is of finite length as an R-module. The second part is easy. □

Remark: The map $\mathbb{F}_p[\epsilon] \to \mathbb{F}_p(t)[\epsilon]$ that takes ϵ to ϵt and $\mathbb{F}_p$ to $\mathbb{F}_p(t)$ does not have a continuous extension to the pro-artinian DP-completions. In fact there is a unique divided power structure on $\mathbb{F}_p(t)\epsilon$ for which $\gamma_p(\lambda\epsilon) = \lambda^p\epsilon$. The kernel of the induced map $\Gamma(\mathbb{F}_p[\epsilon], \mathbb{F}_p[\epsilon]\epsilon) \to \mathbb{F}_p(t)[\epsilon]$ does not have a kernel of finite colength, in fact $\gamma_p^n(\epsilon)$ maps to $t^{p^n}\epsilon$ which are $\mathbb{F}_p$-linearly independent.

Definition 1.3 *A local homomorphism $(\Lambda, \mathfrak{m}_\Lambda) \to (\Lambda_1, \mathfrak{m}_{\Lambda_1})$ is* slightly ramified *if the composite homomorphism $\Lambda \to \hat{\Gamma}(\Lambda_1, \mathfrak{m}_{\Lambda_1})$ is injective. Otherwise the map is* highly ramified.

We illustrate this with some examples.

Lemma 1.4 *A local homomorphism $(\Lambda, \mathfrak{m}_\Lambda) \to (\Lambda_1, \mathfrak{m}_{\Lambda_1})$ of DVRs of mixed characteristic p is slightly ramified if and only if $e_1 < p$, where e_1 is the absolute ramification index e_1 of Λ_1.*

PROOF: This is an easy consequence of the well known fact that Λ_1 has a DP structure on its maximal ideal if and only if $e_1 < p$. □

Lemma 1.5 *Suppose that $\Lambda \to \Lambda_1 \to \Lambda_2$ are local homomorphisms of local rings such that $\Lambda_1 \to \Lambda_2$ induces a finite extension of residue fields. If $\Lambda \to \Lambda_2$ is slightly ramified, then so is $\Lambda \to \Lambda_1$.*

PROOF: By (1.2) the sequence of homomorphisms $\Lambda \to \Lambda_1 \to \Lambda_2$ maps to a sequence $\Lambda \to \hat{\Gamma}(\Lambda_1, \mathfrak{m}_{\Lambda_1}) \to \hat{\Gamma}(\Lambda_2, \mathfrak{m}_{\Lambda_2})$ where the composite $\Lambda \to \hat{\Gamma}(\Lambda_2, \mathfrak{m}_{\Lambda_2})$ is injective. Then $\hat{\Gamma}(\Lambda_2, \mathfrak{m}_{\Lambda_1})$ is also injective. □

Now let $\mathbf{W}$ denote a Cohen ring for the field $\mathbf{k}$ of characteristic p.

Lemma 1.6 *Suppose that the complete local domain R is finite over $\mathbf{W}$, with ramification index e. Suppose that R has normalization $\mathcal{O}$ and that $p = u.\pi^e$, with π a uniformizer in $\mathcal{O}$ and u a unit. Say $\mathfrak{m}_R.\mathcal{O} = \pi^c.\mathcal{O}$. Then $\mathbf{W} \to R$ is slightly ramified if $e < cp$.*

PROOF: Put $S = \mathbf{W} + \pi^c.\mathcal{O}$, a subring of $\mathcal{O}$ containing R. Then $R \to S$ is finite and it is enough to show that $\mathbf{W} \to S$ is slightly ramified. For this, it is enough to show that $(S, \mathfrak{m}_S)$ has a DP structure. Note that $\mathfrak{m}_S = \pi^c.\mathcal{O}$, so that $(S, \mathfrak{m}_S)$ has a DP structure if and only if $(\mathcal{O}, \pi^c.\mathcal{O})$ does. But this holds if and only if $v_p(\pi^c) > 1/p$, which translates as $e < cp$. □

Corollary 1.7 $\mathbf{W}[\![x]\!]/(p^a - x^b)$ *is slightly ramified over* $\mathbf{W}$ *if* $b < ap$.

PROOF: First, assume that a, b are coprime. Then $x = \pi^a$ and $p = \pi^b$. Then $e = b$ and $c = \min(a, b)$.

In general, write $a = ha_1$ and $b = hb_1$, where $h = HCF(a, b)$. Put $R = \mathbf{W}[\![x]\!]/(p^a - x^b)$ and $R_1 = \mathbf{W}[\![x]\!]/(p^{a_1} - x^{b_1})$. Then we have just shown that $\mathbf{W} \to R_1$ is slightly ramified, so that $\mathbf{W} \to R$ is slightly ramified by (1.5). □

Corollary 1.8 *If* $\min(b, d) < ap$*, then* $A := \mathbf{W}[\![x, y]\!]/(p^a - x^b - y^d)$ *is slightly ramified over* $\mathbf{W}$.

PROOF: There are surjections $A \to \mathbf{W}[\![x]\!]/(p^a - x^b)$ and $A \to \mathbf{W}[\![y]\!]/(p^a - y^d)$. Now use (1.7) and (1.5). □

In particular, the E_8 singularities $R = \mathbb{Z}_2[\![x, y]\!]/(2^a - x^b - y^d)$, where $\{a, b, d\} = \{2, 3, 5\}$, are slightly ramified over $\mathbb{Z}_2$, although in each case every DVR that is finite over R and centered at the maximal ideal is highly ramified over $\mathbb{Z}_2$.

2 Smooth foliations

In this section we shall study a condition on rings that will arise as a consequence of assuming the TLP or the DPTLP on a deformation functor. We have in mind applications of these idea to situations outside of those that will appear in the present paper and hence the setting will be more general than would be needed our current purposes.

We are going to use an argument of Zariski, Lipman and Nagata in several slightly distinct situations and the next result is our attempt to extract the essence of their argument in order to be able to apply it to the different situations. Curiously enough our result has no direct relation with the original argument and instead appears as a result on group actions (which is essentially well known but we have not been able to find a reference which would directly apply to our situation).

Lemma 2.1 *Suppose that $\mathcal{C}$ is a category with fibre products, G a group object in $\mathcal{C}$ and $G \times X \to X$ an action of G on $X \in ob(\mathcal{C})$. Assume that we have an equivariant map $s: X \to G$, where G acts on itself by left multiplication. Define Y to be the pullback of s along the identity and $h: G \times Y \to X$ to be the composite of the inclusion $G \times Y \to G \times X$ with the action of G on X. Then h is a G-equivariant isomorphism, where G acts on $G \times Y$ by left multiplication on the first factor and trivially on Y and $s = pr_G \circ h^{-1}$, where $pr_G : G \times Y \to G$ is the projection.*

PROOF: (We shall use set-theoretic notation which can be translated into a proof using only the given structure maps at will.) Define $t: X \to Y$ by $t(x) = s(x)^{-1}x$, which maps into Y as $s(t(x)) = s(s(x)^{-1}x) = s(x)^{-1} \cdot s(x) = e$. We then have the map $(s, t): X \to G \times Y$ which is easily seen to be the inverse to h. □

Corollary 2.2 *With the notation and assumptions of (2.1) assume also that $\mathcal{C}$ is the category of (formal) schemes. Then $h^{-1} \circ pr_Y : X \to Y$ is a quotient by the action of G on X.*

We shall need to generalise the notion of a smooth (height 1) foliation from the case of a smooth variety discussed in [Ek86] to the case of singular varieties. Just as in that case the theory is very different in the case of characteristic zero and positive characteristic and this will soon be apparent. Our first definition however works in any characteristic.

Let $f: X \to S$ be a morphism of finite type of noetherian (formal) schemes. Assume that for every closed point of X the residue field extension of the image of the point in S and that of the point itself is separable (to us a *separable field extension* will be algebraic).

Definition 2.3 *(1) A* smooth foliation *on f is a subsheaf $\mathcal{F}$ of the tangent sheaf $T_{X/S}$ of the sheaf of (continuous) S-derivations of X such that the induced map $\mathcal{F}\bigotimes_{\mathcal{O}_X}\mathbf{k}(x) \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_s + \mathfrak{m}_x^2, \mathbf{k}(x))$ is injective for all points x of X, where $s := f(x)$, and such that $\mathcal{F}$ is closed under commutators. Moreover, if S is over a field of positive characteristic p, then we demand that $\mathcal{F}$ be closed under p'th powers.*

(2) If $Z \hookrightarrow X$ is a closed subscheme then $\mathcal{F}$ is transverse *to Z if for all points $x \in X$ the composite $\mathcal{F}\bigotimes_{\mathcal{O}_X}\mathbf{k}(x) \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_s + \mathfrak{m}_x^2, \mathbf{k}(x)) \to \mathrm{Hom}_{\mathbf{k}(x)}(I/\mathfrak{m}_x I, \mathbf{k}(x))$ is injective, where I is the ideal of Z.*

(3) An S-map $f: X \to Y$ is a quotient *of $\mathcal{F}$ if $\mathcal{F}$ is the relative tangent sheaf of it and $\mathcal{O}_Y$ consists of the $\mathcal{F}$-constants (the sections annihilated by the local sections of $\mathcal{F}$) of $f_*\mathcal{O}_X$.*

Remark: (1) When f is smooth, the injectivity condition simply says that $\mathcal{F}$ is a subbundle of $T_{X/S}$ so this corresponds to the already established notion.

(2) Note that a smooth foliation is transverse to every closed point.

(3) It is not clear that our definition is the right one in the mixed characteristic case. The reason is that we do not know what to do with the condition of being closed under p'th powers. On the one hand it makes no sense in characteristic zero, on the other hand in the mixed characteristic situation, the reduction modulo p of a smooth foliation should be a smooth foliation and hence some condition is needed. One could require that for D in the foliation D^p should be of the form $E + pF$ for p a prime, F a differential operator and E in the foliation but that seems rather artificial.

A similar problem occurs when later in this section we shall consider divided power foliations. Again, if the divided power structure is trivial, i.e., when the divided power ideal is the zero ideal one needs some p-integrability condition. On the other hand, even in positive characteristic p the p-integrability for divided power derivations does not make sense as the p'th power of a divided power derivation is not a divided power derivation. However, our definition turns out to be adequate for the purposes of this paper.

We shall see that, just as in the smooth case, a smooth foliation gives rise to a flat infinitesimal equivalence relation on X. We start with a preliminary result on modules.

Suppose that $(A, \mathfrak{m}, \mathbf{k})$ be a local ring and that M, N are A-modules with N a finitely generated. Assume given an A-homomorphism $\phi : N \to M^* := \mathrm{Hom}_A(M, A)$ and denote by ϕ^* the composite homomorphism $M \to M^{**} \to N^*$.

Lemma 2.4 *Assume that either the map $\bar{\phi} = \phi\bigotimes 1 : N/\mathfrak{m}N \to \mathrm{Hom}_{\mathbf{k}}(M/\mathfrak{m}M, \mathbf{k})$ induced by ϕ is injective or $N = M^*$ and $\bar{\phi}$ is surjective. Then the following are true.*

(1) ϕ is injective and N is free.

(2) M is a direct sum $M_1 \bigoplus M_2$ such that M_1^* is identified with $\phi(N)$, N and M_2 are mutual annihilators and $N/\mathfrak{m}N$ and $M_2/\mathfrak{m}M_2$ are mutual annihilators.

(3) Given elements $m_1, \dots, m_n$ of M_1, they form a basis for M_1 if and only if the elements $\phi^*(m_1), \dots, \phi^*(m_n)$ form a basis of N^*, or if and only if $\phi^*(m_1), \dots, \phi^*(m_n)$ map to elements forming a basis of $\mathrm{Hom}_{\mathbf{k}}(N/\mathfrak{m}N, \mathbf{k})$ under the natural homomorphisms $N^* \to \mathrm{Hom}_{\mathbf{k}}(N/\mathfrak{m}N, \mathbf{k})$.

PROOF: Assume first that $\bar{\phi}$ is injective. Let $n := \dim_{\mathbf{k}} N/\mathfrak{m}N$ and pick elements $m_1, \dots, m_n$ of M such that the composite of $\bar{\phi}$ and the map $\mathrm{Hom}_{\mathbf{k}}(M/\mathfrak{m}M, \mathbf{k}) \to \mathbf{k}^n$ given by evaluation at the residues $\overline{m_i} \in M/\mathfrak{m}M$ is an isomorphism. This gives us a map $A^n \to M$ given by $(r_i) \mapsto \sum_i r_i m_i$. Evaluation at the m_i gives its transpose $M^* \to A^n$ and the composite $N \to M^* \to A^n$ induces an isomorphism upon reduction modulo $\mathfrak{m}$. By Nakayama's lemma it is then surjective and by the projectivity of A^n and Nakayama's lemma again it is an isomorphism. This shows that N is a direct factor of M^* as well as isomorphic to A^n. By construction the basis (n_i) of N thus constructed is dual to $\{m_i\}$ so that the composite $A^n \xrightarrow{(m_i)} M \xrightarrow{(n_i)} A^n$ is the identity and hence the m_i generate a direct summand of M. This shows that M is isomorphic to $M_1 \bigoplus M_2$ in such a way that $N \to M^*$ is the transpose of the projection $M_1 \bigoplus M_2 \to M_1$. Using this description the second statement is clear and the last is obvious.

If instead $N = M^*$ and $\bar{\phi}$ is surjective then we can choose $m_1, \dots, m_n \in M$ which form a basis for $M/\mathfrak{m}M$ and then by assumption find $f_1, \dots, f_n \in M^*$ such that $f_i m_j \equiv \delta_{ij} \bmod \mathfrak{m}$. This gives a map $M \to A^n$ which is surjective by Nakayama's lemma and then an isomorphism as it is split. In particular we have the first situation and the results already obtained apply. □

An immediate consequence of this lemma is that the tangent sheaf is a smooth foliation under conditions weaker than those of the definition.

Proposition 2.5 *Let $f: X \to S$ be a morphism of finite type of noetherian (formal) schemes. If for every point x of X the map $T_{X/S,x} \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_{f(x)} + \mathfrak{m}_x^2, \mathbf{k}(x))$ is surjective, then $T_{X/S}$ is a smooth foliation.*

PROOF: By (2.4) the condition implies that the natural evaluation homomorphism $T_{X/S,x} \bigotimes \mathbf{k}(x) \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_{f(x)} + \mathfrak{m}_x^2, \mathbf{k}(x))$ is an isomorphism. □

Definition 2.6 *The map f is* tangent smooth *when $T_{X/S}$ is a smooth foliation.*

We now show that smooth foliations on a singular variety behave much like smooth foliations on a smooth variety.

Theorem 2.7 *Let $f: X \to S$ be a morphism of finite type of (formal) schemes with S noetherian and $\mathcal{F}$ be a smooth foliation for f transverse to a closed subscheme Z of X.*

(1) Locally $\Omega^1_{X/S}$ splits as a sum $\mathcal{G}' \bigoplus \mathcal{G}''$, where $\mathcal{G}'$ is free with a basis of the form dx_i, $\mathcal{F} = (\mathcal{G}')^*$ and $\mathcal{F}$ is identified with the annihilator of $\mathcal{G}''$. In particular, $\mathcal{F}$ is locally free and locally $T_{X/S}$ splits as $\mathcal{F} \bigoplus (\mathcal{G}'')^*$. Furthermore, at a point of Z the x_i can be chosen to lie in I_Z.

(2) If S is over $\mathbf{Spec}\,\mathbb{Q}$ let $\hat{X}$ be the formal completion of X along Z and $\hat{S}$ that of S along the image of Z. Then the restriction of $\mathcal{F}$ has a quotient $\hat{X} \to Y$ which is formally smooth and the composite $Z \to \hat{X} \to Y$ is an embedding. If $\mathcal{F} = T_{X/S}$ (which is possible precisely when f is tangent smooth) then the structure map $Y \to \hat{S}$ is an immersion.

(3) If S is over $\mathbf{Spec}\,\mathbb{F}_p$ then $\mathcal{F}$ has a quotient $X \to Y$ which is flat and the composite $Z \to X \to Y$ is an embedding. Locally $X \to Y$ has the form $\mathbf{Spec}\,\mathcal{O}_Y[t_1, \dots, t_n]/(t_1^p - f_1, \dots, t_n^p - f_n)$ with the t_i vanishing on Z.

PROOF: For the first part, Lemma 2.4 shows that $\Omega^1_{X/S} = M_1 \bigoplus \mathrm{Ann}(\mathcal{F})$ with $M_1 = \mathcal{F}^*$. Moreover, for all $x \in Z$ there exist $x_1, \dots, x_n \in I_Z$ and $D_1, \dots, D_n \in \mathcal{F}$ such that $D_1, \dots, D_n$ give a basis of $\mathcal{F} \bigotimes \mathbf{k}(x)$ and the matrix $(D_i(x_j) \pmod{\mathfrak{m}_x})$ is an invertible $n \times n$ matrix over $\mathbf{k}(x)$. This completes the proof of (2.7 1).

Suppose that we are over $\mathbf{Spec}\,\mathbb{Q}$. We use a slight modification of the proof of a result of Zariski, Lipman and Nagata ([M86, p. 230]). Since quotients are unique, when they exist, it is enough to work locally on Z. By assumption and (1) we can, on an affine neighbourhood $\mathbf{Spec}\,A$ of a point $x \in Z$, find a basis $D_1, \dots, D_n$ of $\mathcal{F}$ and functions $x_1, \dots, x_n$ vanishing on Z such that $D_i x_j = \delta_{ij}$. This implies that $[D_i, D_j]x_k = 0$, so that $[D_i, D_j] \in \mathrm{Ann}(\mathcal{G}')$. Since the vector fields $[D_i, D_j]$ lie in $\mathcal{F}$, by assumption, and $\mathcal{F} = \mathrm{Ann}(\mathcal{G}'')$, it follows that $[D_i, D_j] = 0$ for all i and j.

We write $\widehat{X} = \mathbf{Specf}\,R$ and $\widehat{S} = \mathbf{Specf}\,\Lambda$. Let $\hat{\mathbf{G}}$ denote the formal group $\widehat{\mathbf{G}}^n_a = \mathbf{Specf}\,\Lambda[\![t_1, \dots, t_n]\!]$, with formal co-multiplication $t_i \mapsto t_i \hat{\otimes} 1 + 1 \hat{\otimes} t_i$. Then the (continuous extensions of the) derivations D_i define an action of the formal group $\widehat{\mathbf{G}}$ on $\mathbf{Specf}\,R$ by the formal co-action

$$r \mapsto \sum_{\alpha \in \mathbb{N}^n} \frac{t^\alpha}{\alpha!} D^\alpha(r),$$

That this is a co-action follows from the fact that $[D_i, D_j] = 0$. We also define a map $s\colon \mathbf{Specf}\,R \to \widehat{\mathbf{G}}^n_a$ given by $t_i \mapsto x_i$. The condition that $D_i x_j = \delta_{ij}$ is then precisely that s is equivariant and we may now apply (2.1).

Finally, assume that $\mathcal{F} = T_{X/S}$. Then the surjectivity of $\Lambda \to C$, where $C \subset R$ is the ring of $\mathcal{F}$-invariants, follows because modulo $(x_1, \dots, x_n)$ it is surjective.

The characteristic p case is done similarly. We work locally on X and get D_i and x_j in the same way, as well as the relation $[D_i, D_j] = 0$. Moreover, D_i^p is a linear combination of the D_j which gives the relation $D_i^p = 0$. The formula

$$r \mapsto \sum_{0 \le \alpha_i < p} \frac{t^\alpha}{\alpha!} D^\alpha(r),$$

where the t_i are parameters on $\alpha_p^n = \mathbf{Spec}\,\mathbf{k}[t_1, \dots, t_n]/(t_1^p, \dots, t_n^p)$, defines an action of α_p^n on X. This time however the x_i do not necessarily give a map to α_p^n as we may not have $x_i^p = 0$. Instead we consider the α_p^n-torsor $\mathbf{Spec}\,\mathbf{k}[r_1, \dots, r_n] \to \mathbf{Spec}\,\mathbf{k}[s_1, \dots, s_n]$ given by $s_i \mapsto r_i^p$ and the Lie algebra of α_p^n acting as $\partial/\partial r_i$. Then $X \to \mathbf{Spec}\,\mathcal{O}_S[r_1, \dots, r_n]$ given by $r_i \mapsto x_i$ is an equivariant map and as α_p^n acts freely on $\mathbf{Spec}\,\mathcal{O}_S[r_1, \dots, r_n]$ it does so on X. As α_p^n is a finite group scheme there is no problem with taking the quotient and we have a quotient map $X \to Y$ and an induced map $Y \to \mathbf{Spec}\,\mathcal{O}_S[s_1, \dots, s_n]$. Being a map between group torsors this mean that X is the fibre product of $Y \to \mathbf{Spec}\,\mathcal{O}_S[s_1, \dots, s_n]$ and $\mathbf{Spec}\,\mathcal{O}_S[r_1, \dots, r_n] \to \mathbf{Spec}\,\mathcal{O}_S[s_1, \dots, s_n]$ which gives the desired result. Again the claim made about Z is clear. □

We can apply the theorem to the case where $T_{X/S}$ is a smooth foliation. In that case we also get a result in the mixed characteristic case.

Corollary 2.8 *Let $f\colon X \to S$ be a dominant morphism of finite type of (formal) schemes with S noetherian.*

(1) If f is smooth then f is tangent smooth.

(2) $\Omega^1_{X/S}$ is locally free if and only if f is tangent smooth.

(3) If S is over $\mathbf{Spec}\,\mathbb{Q}$ and f is tangent smooth then f is smooth.

(4) If S is over $\mathbb{F}_p$ then f is tangent smooth if and only if for every point $x \in X$ and $s := f(x)$ there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{S,s}[\![t_1, \dots, t_n]\!]/(f_i)$, where $f_i(t_1, \dots, t_n) = g_i(t_1^p, \dots, t_n^p)$ for some $g_i \in \widehat{\mathcal{O}}_{S,s}[\![t_1, \dots, t_n]\!]$.

(5) If $\mathcal{O}_S$ is torsion-free and f is tangent smooth then f is smooth.

PROOF: (3) and (4) follow directly from the theorem and (2.5) together (in the characteristic zero case) with the fact that a dominant immersion is étale.

For the last part, we may localize X and S and then complete, to get $X = \mathbf{Specf}\, R$ and $S = \mathbf{Specf}\, \Lambda$. Then choose $t_1, \ldots, t_n$ in the maximal ideal $\mathfrak{m}_R$ of R that map to a basis of the cotangent space $t^*_{R/\Lambda} = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda)$ and define a map $\mathcal{O} := \Lambda[\![T_1, \ldots, T_n]\!] \xrightarrow{\pi} R$ with $\pi(T_i) = t_i$. Then π is surjective.

Since Λ is torsion-free, there is an embedding $\Lambda \to K$, where K is a finite direct product of characteristic zero fields. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\pi} & R \\ \downarrow & & \downarrow \\ K[\![T_1, \ldots, T_n]\!] & \xrightarrow{\pi_1} & R\widehat{\otimes}_\Lambda K, \end{array}$$

where $R\widehat{\otimes}_\Lambda K$ is the tensor product completed in the $(t_1, \ldots, t_n)$-adic topology. Since π_1 is an isomorphism, by (3), and $\mathcal{O} \to K[\![T_1, \ldots, T_n]\!]$ is injective, it follows that π is injective. $\square$

We have put the positive characteristic case alongside the characteristic zero and mixed characteristic case but in actuality it is rather different and we finish this section with a discussion special to that case.

We shall say that a local ring $(R, \mathfrak{m})$ with $pR = 0$, p a prime, is of *height* 1 if $f^p = 0$ for all $f \in \mathfrak{m}$. We shall also use the notation $\mathfrak{m}^{(p)}$ for the ideal generated by the p'th powers of elements of $\mathfrak{m}$. It is then clear that $R/\mathfrak{m}^{(p)}$ is the largest height 1 quotient of R. We shall also say that R is *formally height* 1*-smooth* if it has the infinitesimal lifting property with respect to maps into height 1 local rings.

Proposition 2.9 *Let p be a prime and $(R, \mathfrak{m})$ a local noetherian ring with $pR = 0$ and perfect residue field $\mathbf{k}$. The following conditions are equivalent:*

(i) R is formally height 1-smooth.

(ii) $R/\mathfrak{m}^{(p)}$ is of the form $\mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)}$ for some n.

PROOF: This is completely analogous to the similar relation between the infinitesimal lifting property and formal smoothness:

We may assume that R is complete and then, as $\mathbf{k}$ is perfect, R contains $\mathbf{k}$ and is thus the quotient of $\mathbf{k}[\![t_1, \ldots, t_n]\!]$ with n minimal. We then use the height 1 lifting property for the map $\mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)} \to \mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^2$ to get a map $R \to \mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)}$. This induces a map $R/\mathfrak{m}^{(p)} \to \mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)}$ whose composite with $\mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)} \to R/\mathfrak{m}^{(p)}$ is an automorphism of $\mathbf{k}[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^{(p)}$. $\square$

As mentioned in the introduction, when applying our results to deformation of Calabi-Yau varieties we shall want to restrict ourselves to deformations over elements of C_Λ which have a divided power structure on their maximal ideals. As should perhaps be no surprise the results of this section become more uniform (and hence more like the results in characteristic 0) if we modify our arguments by using divided powers.

We need to recall some facts about continuous derivations and differentials for continuous maps of topological divided power pairs $(\Lambda, J) \to (R, I)$. A (continuous) divided power derivation consists of a (topological) module M over R and a (continuous) Λ-linear map $D\colon R \to M$ such that $D(rs) = rD(s) + sD(r)$ and $D\gamma_n(i) = \gamma_{n-1}(i)D(i)$ for all $r, s \in R$ and $i \in I$. There then is a universal (continuous) divided power derivation $d\colon R \to \Omega^{1,DP}_{R/\Lambda}$. It has the property that if $\mathfrak{m}_R \supseteq I$ is a maximal ideal that induces a maximal ideal $\mathfrak{m}_\Lambda$ of Λ such that the

residue field extension is separable then $\Omega^{1,DP}_{R/\Lambda} \bigotimes R/\mathfrak{m}_R$ is (canonically) isomorphic to $\mathfrak{m}_R/\mathfrak{m}_R^2 + I^{[2]} + \mathfrak{m}_\Lambda$.

Definition 2.10 *A map* $(\Lambda, J) \to (R, I)$ *as above is* of finite type *if it arises from taking DP envelopes, or the pro-artinian completions of these, of an essentially finite type homomorphism.*

Lemma 2.11 *If* $(\Lambda, J) \to (R, I)$ *is a continuous map of topological divided power pairs, then* $\Omega^{1,DP}_{R/\Lambda}$ *is a finitely generated* R*-module.*

For a (continuous) map of (topological) divided power pairs $(\Lambda, J) \to (R, I)$ we let $\mathrm{Der}^{DP}_{R/\Lambda}$, which we shall also denote $T^{DP}_{R/\Lambda}$, consist of the (continuous) divided power derivations of R. The pair $(R[\epsilon], I[\epsilon])$ has a unique (topological) divided power structure extending that of (R, I) and having $\gamma_n(\epsilon) = 0$ if $n > 1$. It is then easy to see that a (continuous) divided power map $R \to R[\epsilon]$ which is the identity modulo ϵ is the same thing as a (continuous) divided power derivation of R.

There is now an obvious extension of the notions of a smooth foliation and a tangent smooth map to the divided power case. We express this in a global context.

Definition 2.12 *Suppose that* $(X, Y) \to (S, T)$ *is a divided power map of divided power pairs of (formal) schemes and that the residue fields of all closed points are perfect.*

(1) A smooth DP foliation *(transverse to* Y*) on the map is an* $\mathcal{O}_X$*-submodule* $\mathcal{F}$ *of* $T^{DP}_{X/S}$ *which is closed under commutators such that for all closed points* $x \in X \setminus Y$ *the natural map* $\mathcal{F}/\mathfrak{m}_x\mathcal{F} \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2 + I^{[2]} + \mathfrak{m}_{f(x)}, \mathbf{k}(x))$ *is injective and for each closed point* $y \in Y$ *the induced map* $\mathcal{F} \bigotimes \mathbf{k}(y) \to \mathrm{Hom}_{\mathbf{k}(y)}(I_Y \bigotimes \mathbf{k}(y), \mathbf{k}(y))$ *is injective.*

(2) The map is DP tangent smooth *if* $T^{DP}_{X/S}$ *is a smooth DP foliation.*

There are immediate analogues of Proposition 2.5 and Theorem 2.7 1. We give these together in the next lemma. The proofs are identical, so omitted.

Lemma 2.13 *Suppose that* $f\colon (X, Z) \to (S, T)$ *be a finite type divided power morphism of (formal) divided power schemes.*

(1) If for any point x *of* X*, the map* $T^{DP}_{X/S,x} \to \mathrm{Hom}_{\mathbf{k}(x)}(\mathfrak{m}_x/(\mathfrak{m}_{f(x)} + \mathfrak{m}_x^2 + I_Z^{[2]}), \mathbf{k}(x))$ *is surjective, then* $T_{X/S}$ *is a smooth DP foliation.*

(2) Suppose that $\mathcal{F}$ *is a smooth DP foliation for* f *transverse to* Z*. Then, locally on* X*, we have* $\Omega^{1,DP}_{X/S} = \mathcal{G}' \bigoplus \mathcal{G}''$*, where* $\mathcal{G}'$ *is free with a basis of the form* $\{dx_i\}$*,* $\mathcal{F} = (\mathcal{G}')^*$ *and* $\mathcal{F}$ *is identified with the annihilator of* $\mathcal{G}''$*. In particular,* $\mathcal{F}$ *is locally free and locally* $T^{DP}_{X/S}$ *splits as* $\mathcal{F} \bigoplus (\mathcal{G}'')^*$*. Furthermore, at a point of* Z *the* x_i *can be chosen to lie in* I_Z*.*

Now fix a complete DP local ring Λ and consider the category $\mathcal{C}$ of local pro-DP-artin Λ-algebras R with a section $R \to \Lambda$ and the further property that $\ker(R \to \Lambda)$ is (topologically) DP-nilpotent. This category has finite sums, denoted by $\hat{\otimes}$, whose construction is left to the reader.

Lemma 2.14 *If* $A \in \mathcal{C}$*, the DP envelope* Γ *of the pair* $(A[t_1, \ldots, t_n], (t_1, \ldots, t_n))$ *has a DP structure on the maximal ideal* $\mathfrak{m}$ *generated, as a DP ideal, by* $(\mathfrak{m}_A, t_1, \ldots, t_n)$ *and the ring* $A\langle\langle t_1, \ldots, t_n \rangle\rangle$ *constructed as the pro-DP-artin completion of* Γ *at* $\mathfrak{m}$ *lies in* $\mathcal{C}$*.*

PROOF: Denote by M the free A-module with basis $\{t_1, \ldots, t_n\}$. Then $\Gamma = \bigoplus \Gamma^n(M)$ and $A\langle\langle t_1, \ldots, t_n \rangle\rangle = \prod \Gamma^n(M)$. Now the result, in particular the fact that $A\langle\langle t_1, \ldots, t_n \rangle\rangle$ is $(t_1, \ldots, t_n)$-DP-adically separated, is clear. ◻

Theorem 2.15 *(1) Suppose that $\mathcal{F}$ is a smooth DP foliation for the object $\Lambda \to R$ of $\mathcal{C}$, transverse to the given section* $\mathbf{Specf}\,\Lambda$ *of* $\mathbf{Specf}\,R$. *Put $\Lambda' = R^{\mathcal{F}}$, the ring of $\mathcal{F}$-invariants. Then Λ' is in $\mathcal{C}$ and R is isomorphic to $\Lambda'\langle\langle t_1, \dots, t_n\rangle\rangle$.*

(2) Assume that $\mathbf{k}$ is a perfect field of positive characteristic p and let $(\Lambda, \mathfrak{m}_\Lambda)$ be a local $\mathbf{k}$-algebra whose residue field is a finite extension of $\mathbf{k}$. Assume that its divided power hull is DP tangent smooth over $\mathbf{k}$. Then it is height 1 smooth.

(3) Assume that $\Lambda_0 \xrightarrow{f} R \to \Lambda_1$ are local maps of noetherian local rings, that the induced extensions of residue fields are finite, that Λ_0 is torsion-free and that $\Lambda_0 \to \Lambda_1$ is slightly ramified. Suppose that $\hat{f} : \widehat{\Lambda}_0 \to \widehat{R}$ is the induced homomorphism between the pro-DP-artin completions of the DP envelopes of the first two rings. If $\hat{f}$ is DP tangent smooth, then f is formally smooth.

PROOF: The proof of the first statement is essentially identical to what has already been proven in characteristic zero; we get commuting generators $D_1, \dots, D_n$ of $\mathcal{F}$ and topological DP generators $x_1, \dots, x_n$ of $\ker(R \to \Lambda)$ such that $D_i(x_j) = \delta_{ij}$. The difference is that the formal group $\widehat{G}_a^n$ must be replaced by the commutative formal group $\widehat{G} = \mathbf{Specf}\,\widehat{\Gamma}$, where $\widehat{\Gamma} = \Lambda\langle\langle t_1, \dots, t_n\rangle\rangle$ and the formal co-multiplication is $t_i \mapsto t_i \hat{\otimes} 1 + 1 \hat{\otimes} t_i$. There is a formal co-action $\widehat{R} \to \widehat{R} \mathbin{\hat{\otimes}} \widehat{\Gamma}$ given by

$$r \mapsto \sum D^\alpha(r) \mathbin{\hat{\otimes}} \gamma_\alpha(t),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\gamma_\alpha(t) = \gamma_{\alpha_1}(t_1) \cdots \gamma_{\alpha_n}(t_n)$. (The $(t_1, \dots, t_n)$-DP-adic separation is needed to ensure the convergence of this formula.) As in characteristic zero, there is, because $\widehat{R}$ is (x)-DP-adically separated, a $\widehat{G}$-equivariant map $s\colon \mathbf{Specf}\,\widehat{R} \to \widehat{G}$ given by $t_i \mapsto x_i$, and we conclude by (2.1) and (2.2).

As for (2) as the residue field of Λ is also perfect we may assume that it equals $\mathbf{k}$. we may choose elements basis $t_1, \dots, t_n \in \mathfrak{m}_\Lambda$ such that they form a basis for $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$. We then have a surjective ring homomorphism $\mathbf{k}[[t_1, \dots, t_n]] \to \Lambda$. Let R be the divided power hull of Λ. Using the notations of the first part we then have $R \xrightarrow{\sim} \Lambda'\langle\langle t_1, \dots, t_n\rangle\rangle$ and we have a $\mathbf{k}$-homomorphism $\Lambda' \to k$. Composing we get maps $\mathbf{k}[[t_1, \dots, t_n]]/\mathfrak{m}^{(p)} \to \Lambda/\mathfrak{m}_\Lambda^{(p)} \to \mathbf{k}\langle\langle t_1, \dots, t_n\rangle\rangle/\mathfrak{m}^{[p]}$ and as the composite is an isomorphism, the first map is injective and as it is also surjective it is an isomorphism. We then conclude by Proposition 2.9.

For the last part, we can assume that the rings are complete. We let $\widehat{\Lambda}_i$ denote the pro-DP-artin completion of the DP hull of $(\Lambda_i, \mathfrak{m}_{\Lambda_i})$. Moreover, after replacing Λ_1 by its image in $\widehat{\Lambda}_1$, we can assume that Λ_1 injects into $\widehat{\Lambda}_1$. Of course, Λ_0 injects into Λ_1.

Choose $x_1, \dots, x_n \in \mathfrak{m}_R$ that map to a basis of the cotangent space $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\Lambda_0})$. Then there is a surjection $\pi : \mathcal{O}_0 := \Lambda_0[[t_1, \dots, t_n]] \to R$ with $t_i \mapsto x_i$. Say $J = \ker \pi$; then $J \subset \mathfrak{m}_{\mathcal{O}_0}^2 + \mathfrak{m}_{\Lambda_0}\mathcal{O}_0$.

Put $R_1 = R \mathbin{\widehat{\otimes}}_{\Lambda_0} \Lambda_1$ and $\mathcal{O}_1 = \mathcal{O}_0 \mathbin{\widehat{\otimes}}_{\Lambda_0} \Lambda_1$. Write x_i for $x_i \mathbin{\widehat{\otimes}} 1$ and t_i for $t_1 \mathbin{\widehat{\otimes}} 1$. Let $\widehat{\mathcal{O}}_1$, resp., $\widehat{R}_1$, be the (t)-DP-nilpotent, resp. (x)-DP-nilpotent, pro-artin DP-completion of $\mathcal{O}_1$, resp. R_1. Then the sequence $\Lambda_1 \to \mathcal{O}_1 \to R_1 \to \Lambda_1$ maps to a sequence $\widehat{\Lambda}_1 \to \widehat{\mathcal{O}}_1 \to \widehat{R}_1 \to \widehat{\Lambda}_1$. Then (1) (more precisely, its proof) shows that $\hat{\pi}_1 : \widehat{\mathcal{O}}_1 \to \widehat{R}_1$ is an isomorphism.

Consider the square

$$\begin{array}{ccc} \mathcal{O}_1 & \xrightarrow{\pi_1} & R_1 \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_1 & \xrightarrow{\hat{\pi}_1} & \widehat{R}_1. \end{array}$$

Now $\mathcal{O}_1$ is, as a Λ_1-module, the direct product $\prod S^m(M)$, where M is the free Λ_1-module with basis $\{t_1, \dots, t_n\}$, while $\widehat{\mathcal{O}}_1$ is, as a $\widehat{\Lambda}_1$-module, the direct product $\prod \Gamma^m(M \otimes_{\Lambda_1} \widehat{\Lambda}_1)$. Since Λ_0 is torsion-free, the natural map $\mathcal{O}_1 \to \widehat{\mathcal{O}}_1$ given by

$t^m \mapsto m!\gamma_m(t)$ is injective. Since $\hat{\pi}_1$ is an isomorphism, it follows that π_1 is injective, so an isomorphism.

Next, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & J & \to & \mathcal{O}_0 & \xrightarrow{\pi} & R & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J_1 & \to & \mathcal{O}_1 & \xrightarrow{\pi_1} & R_1 & \to & 0. \end{array}$$

Since π_1 is injective and $\Lambda_0 \to \Lambda_1$ is injective, $\mathcal{O}_0 \to \mathcal{O}_1$ is injective, and so, by the snake lemma, $J = 0$. $\square$

3 The tangent lifting property

In preparation for the study of deformations of Calabi-Yau manifolds we shall say that $\overline{F}$ fulfills the *divided power tangent lifting property* (abbreviated to DPTLP) if

$$\overline{F}(A[\epsilon]) \to \overline{F}(A) \times \overline{F}(\mathbf{k}[\epsilon])$$

is surjective for all $A \in C_\Lambda$ for which the pair $(A, \mathfrak{m}_A)$ admits a divided power structure. Note that under condition H_2 on $\overline{F}$ this is equivalent to $\overline{F}(A[\epsilon]) \to \overline{F}(A \times_{\mathbf{k}} \mathbf{k}[\epsilon])$ being surjective.

Proposition 3.1 *Let Λ be complete local ring with residue field $\mathbf{k}$, a perfect field, and assume that $\overline{F}: C_\Lambda \to \underline{\mathrm{Sets}}$ is a covariant functor from the category of local artinian Λ-algebras with residue field $\mathbf{k}$ to the category sets such that $\overline{F}(\mathbf{k})$ is a point (we call this condition H_0) and $\overline{F}$ fulfills the conditions (H_1-H_3) of Schlessinger (cf. [Sch68, Thm. 2.11]). Let R be a hull for $\overline{F}$.*

(1) Let $\pi: \overline{H} \to \overline{G}$ be a formally smooth map between functors $C_\Lambda \to \underline{\mathrm{Sets}}$ that both fulfill conditions H_0 and H_2. Then $\overline{H}$ fulfills the (divided power) tangent lifting property precisely when $\overline{G}$ does. In particular $\overline{F}$ fulfills the (divided power) tangent lifting property precisely when h_R does (where $h_R(A) = \mathrm{Hom}_\Lambda(R, A)$).

(2) h_R fulfills the tangent lifting property precisely when R is tangent smooth.

(3) h_R fulfills the divided power tangent lifting property precisely when $\hat{\Gamma}(R, \mathfrak{m}_R)$ is divided power tangent smooth.

Remark: It will be clear from the proof that other conditions than H_2 would also give (1), for instance that $\overline{H}(\mathbf{k}[\epsilon]) \to \overline{G}(\mathbf{k}[\epsilon])$ is a bijection.

PROOF: Assume that $\overline{G}$ has the tangent lifting property. Suppose $A \in C_\Lambda$, and $\phi \in \overline{H}(A \times_{\mathbf{k}} \mathbf{k}[\epsilon])$. We must, by the comment made at the beginning of this section, find $\Phi \in \overline{H}(A[\epsilon])$ with $\Phi \mapsto \phi$. Let $\xi \in \overline{G}(A \times_k \mathbf{k}[\epsilon])$ be the image under π of ϕ. By the tangent lifting property there is a $\Xi \in \overline{G}(A[\epsilon])$ mapping to ξ. By the formal smoothness of π and the fact that $A[\epsilon] \to A \times_{\mathbf{k}} \mathbf{k}[\epsilon]$ is surjective there is a $\Phi \in \overline{H}(A[\epsilon])$ mapping to Ξ and ϕ. The converse is similar (but easier). The last statement of (1) follows from the fact that as R is a hull there is a formally smooth map $h_R \to \overline{F}$.

Turning to (2) assume that h_R has the tangent lifting property. Write $R = \varprojlim R_\alpha$ as a limit of local artinian rings with surjective transition maps inducing an isomorphisms on cotangent spaces. Given $v \in t_R$ and an index α there is a Λ-derivation $D: R \to R_\alpha$ inducing the given v on cotangent spaces. The set of such derivations is a coset for the kernel of the map $\mathrm{Hom}_R(\Omega^1_{R/\Lambda}, R_\alpha) \to \mathrm{Hom}_{\mathbf{k}}(\mathfrak{m}_R/\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda, \mathbf{k})$. That kernel is of finite length as R-module so by linear compactness there is an element D of $\varprojlim \mathrm{Hom}_R(\Omega^1_{R/\Lambda}, R_\alpha)$ mapping to v. The D corresponds to an inverse system of derivations $D_\alpha: R \to R_\alpha$ and passing to the limit gives a derivation $R \to R$. The converse is clear.

In the divided power case we let $(\hat{R}, \hat{\mathfrak{m}}) \in \widehat{C}_\Lambda$ be the completed divided power hull of $(R, \mathfrak{m})$. By definition we have $\hat{R} = \varprojlim R_\alpha$, where R_α form an inverse system of divided power local artinian algebras (and divided power maps between) inducing an isomorphisms on (divided power) cotangent spaces. Fix a $v \in t^{DP}_{\hat{R}} = t_R$. For any given α we may consider the composite $R \to \hat{R} \to R_\alpha$ and by the divided power tangent lifting property there is a ring homomorphism $R \to R_\alpha[\epsilon]$ inducing v on cotangent spaces. We may give $R_\alpha[\epsilon]$ the divided power structure which is the given one on the maximal ideal of R_α and for which all higher divided powers of ϵ are zero. By the universal property of $\hat{R}$ this extends to a divided power map $\hat{R} \to R_\alpha[\epsilon]$ reducing modulo ϵ to the given one. This then corresponds to a divided power derivation $\hat{R} \to R_\alpha$ inducing v on cotangent spaces. The rest of the argument is then the same as for (2). □

Combining this result with those of the previous section we get Theorem A.

PROOF of Theorem A: The result follows from (3.1) and Corollary 2.8 together with the observations that in the positive characteristic case, if some $f_i \neq 0$ then it is a non-zero nilpotent element of $\mathbf{k}[[t_1, \ldots, t_n]]/(f_i^p)$ whose support is all of $\mathbf{Spec}\,\mathbf{k}[[t_1, \ldots, t_n]]/(f_i^p)$ and that in the mixed characteristic case the existence of Λ_1 forces $\mathbf{Specf}\,R \to \mathbf{Specf}\,\mathbf{W}$ to be dominant. □

PROOF of Theorem B: This is a direct corollary of Proposition 3.1. □

4 Deformation of Calabi-Yau varieties

The results of the previous section can now be applied to the deformation of varieties and in particular to Calabi-Yau varieties.

Proposition 4.1 *Let X be a proper smooth variety over a perfect field $\mathbf{k}$ and let Λ be a complete local ring with residue field $\mathbf{k}$. Suppose that for any deformation $\mathcal{X} \to \mathbf{Spec}\,A$ of X over an algebra $A \in C_\Lambda$, the reduction map $H^1(\mathcal{X}, T_{\mathcal{X}/A}) \to H^1(X, T_{X/\mathbf{k}})$ is surjective. Then the functor of deformations of X over elements of C_Λ has the tangent lifting property.*

PROOF: The proposition follows directly from (0.1) and the usual obstruction theory. □

Remark: In the proof of [M86, p. 230] a simultaneously weaker and stronger result is obtained. It can be reformulated as saying that if R is a complete local ring in characteristic zero and the image of $T_R \to \mathrm{Hom}_{R/\mathfrak{m}_R}(\mathfrak{m}_R/\mathfrak{m}_R^2, R/\mathfrak{m}_R)$ has dimension k, then R is isomorphic to $S[[t_1, \ldots, t_k]]$ for some complete subring S of R. (Similar results can be shown in positive and mixed characteristic.) Applied to the deformation hull R of a smooth and proper variety (still in characteristic zero) it shows that if $\mathbf{k}$ is the dimension of the image of $H^1(\mathcal{X}, T_{\mathcal{X}}) \to H^1(X, T_X)$, where $\mathcal{X}$ is a miniversal deformation, then R is of the form $S[[t_1, \ldots, t_k]]$. Not much seems to be known about this invariant. Can it, for instance, be trivial for a non-rigid surface X?

The application in characteristic zero of this result (or the corresponding result for the T^1-lifting property) to Calabi-Yau varieties is through two facts. First that any deformation of a smooth proper variety with ω_X trivial has a trivial relative ω and second that the tangent lifting property is true as it comes down to a lifting property of Hodge cohomology which is always true. In positive characteristic neither of these facts remain true. An α_2-Enriques surface which has trivial ω deforms to a $\mathbb{Z}/2$-Enriques surface whose ω has order 2. Similarly Hodge cohomology does not always have the lifting property (though we do not know of any examples where

the ω stays trivial in the family). However, if one defines Calabi-Yau varieties (as does Hirokado, cf. [Hi99]) by the condition that ω_X be trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ then any infinitesimal deformation of a Calabi-Yau variety has trivial relative ω in any characteristic. Still it is not clear whether under that assumption the tangent lifting property is automatic. Hence rather than formulating a highly conditional result we will give a weaker result that still is true under rather weak conditions and is the most direct generalisation of the arguments used in characteristic zero. (For the sake of completeness as well as comparison we also give the characteristic zero result.)

Proposition 4.2 *Let X be a smooth and proper purely n-dimensional variety over a perfect field $\mathbf{k}$ with ω_X trivial. Let F be the deformation functor of X over some complete local ring Λ with residue field $\mathbf{k}$. When $\operatorname{char}\mathbf{k} = p > 0$ assume furthermore that $(\Lambda, \mathfrak{m}_\Lambda)$ has been given a divided power structure compatible with the canonical map $\mathbf{W}(\mathbf{k}) \to \Lambda$ and the unique divided power structure on $(\mathbf{W}, p\mathbf{W})$.*

(1) If $\operatorname{char}\mathbf{k} = 0$ then F has the tangent lifting property.

(2) If $\operatorname{char}\mathbf{k} = p > 0$, $p\Lambda = 0$, and $\dim_{\mathbf{k}} H^n_{DR}(X/\mathbf{k}) = \sum_{i=0}^n h_X^{in-i}$ then F has the divided power tangent lifting property.

(3) If $\operatorname{char}\mathbf{k} = p > 0$ and $b_n(X) = \sum_{i=0}^n h_X^{in-i}$ then F has the divided power tangent lifting property.

PROOF: In all cases we want to prove first that the relative ω remains trivial over deformations of X over some rings of C_Λ. If so, then for a deformation $\mathcal{X} \to S$ over one of these rings $H^1(\mathcal{X}, T_{\mathcal{X}/S})$ equals $H^1(\mathcal{X}, \Omega^{n-1}_{\mathcal{X}/S})$ and then we want to prove that $H^1(\mathcal{X}, \Omega^{n-1}_{\mathcal{X}/S}) \to H^1(X, \Omega^{n-1}_{X/\mathbf{k}})$ is surjective. Both would follow if we could show that $H^i(\mathcal{X}, \Omega^{n-i}_{\mathcal{X}/S})$ commutes with base change for all $0 \le i \le n$. For the first part this is [De68, Thm. 5.5] for all rings in C_Λ.

The only case where characteristic zero is used in [loc. cit.] is for the degeneration of the Hodge to de Rham spectral sequence and the fact that $H^n_{DR}(-/-)$ commutes with base change. The only consequence of the degeneration that is used is exactly that $\dim_{\mathbf{k}} H^n_{DR}(X/\mathbf{k}) = \sum_{i=0}^n h_X^{in-i}$. On the other hand, let us recall some results from crystalline cohomology: If $A \in C_\Lambda$ and $\mathcal{X} \to \mathbf{Spec}\, A$ is a deformation of X, and if $(A, \mathfrak{m}_A)$ has a divided power structure compatible with that of Λ then $R\Gamma(\mathcal{X}, \Omega^{\cdot}_{\mathcal{X}/A})$ is isomorphic to $R\Gamma(X/\mathbf{W})\bigotimes^L_{\mathbf{W}} A$ resp. to $R\Gamma(X, \Omega^{\cdot}_{X/\mathbf{k}})\bigotimes^L_{\mathbf{k}} A$ if $p\Lambda = 0$ (cf. [Be74, Cor. V:3.5.7]). Consider first the case when $p\Lambda = 0$. Then using the fact that the $H^*_{DR}(X/\mathbf{k})$ are flat $\mathbf{k}$-modules (as all $\mathbf{k}$-modules are) we get that $H^*_{DR}(\mathcal{X}/A) = H^*_{DR}(X/\mathbf{k})\bigotimes_{\mathbf{k}} A$ and as also $\dim_{\mathbf{k}} H^n_{DR}(X/\mathbf{k}) = \sum_{i=0}^n h_X^{in-i}$ Deligne's argument applies. In the general case, the condition $b_n(X) = \sum_{i=0}^n h_X^{in-i}$ is equivalent to $H^i_{crys}(X/\mathbf{W})$ being torsion free for $i = n, n+1$ and $\dim_{\mathbf{k}} H^n_{DR}(X/\mathbf{k}) = \sum_{i=0}^n h_X^{in-i}$. The first part implies that $H^n_{DR}(\mathcal{X}/A) = H^n_{crys}(X/\mathbf{W})\bigotimes_{\mathbf{W}} A$ so again Deligne's argument applies. □

We have now enough results to prove Theorem C.

PROOF of Theorem C: This is proved by combining (4.2), (3.1) and specifically Theorem B for the case of positive and mixed characteristic deformations. □

We do not know of any example of a Calabi-Yau variety fulfilling our conditions, yet having an obstructed deformation space. We shall finish this section by a global result that shows that such examples would have to be quite isolated.

Proposition 4.3 *(1) Let $\mathbf{k}$ be a perfect field of positive characteristic p, S a connected $\mathbf{k}$-scheme of finite type and $f\colon X \to S$ a smooth and proper morphism of pure dimension n that is versal at all of the points of S. Assume that for each $s \in S$ $\dim_{\mathbf{k}(s)} H^n_{DR}(X_s/\mathbf{k}(s)) = \sum_{i+j=n} h^{ij}(X_s)$, that $\dim_{\mathbf{k}(s)} H^n_{DR}(X_s/\mathbf{k}(s))$ is independent of s, and that there is one point $s \in S$ for which ω_{X_s} is trivial. Then S is either a smooth $\mathbf{k}$-variety or S is non-reduced at all its points.*

(2) Suppose instead that S is a $\mathbf{W}(\mathbf{k})$-scheme everywhere versal over $\mathbf{W}$ and that $\overline{S} := S \bigotimes_{\mathbf{W}} \mathbf{k}$ is connected. If $b_n(X_s) = \sum_{i+j=n} h^{i,j}(X_s)$ for all $s \in S$, ω_{X_s} is trivial for some $s \in S$, X_t has a formal lifting over a torsion-free base for some $t \in S$ and S is somewhere smooth, then S is smooth over $\mathbf{W}$.

PROOF: For the first part, we begin by noting that the functions $s \mapsto h^{in-i}_{X_s}$ are constant. For this it is enough to show that they take the same value under a specialisation but they rise under specialisation under semi-continuity but by assumption their sum is constant. From this it follows in particular that sections of ω_{X_s} lift to a generisation so that the set of points for which ω_{X_s} is trivial is open. As it is also closed it equals S by assumption. Hence we get also that $s \mapsto \dim_{\mathbf{k}(s)} H^1(X_s, T^1_{X_s})$ is constant on S.

Assume now that S is reduced somewhere. Then there is an irreducible component Z of S meeting the smooth locus $\mathrm{Reg}(S)$ of S; since S is connected, it is enough to show that S is smooth at every point s of Z.

Suppose that $g\colon Y \to D$ is miniversal at $d \in D$ with $Y_d \cong X_s$. Then, after shrinking (in the étale topology) S and D appropriately, there is a smooth surjective morphism $S \to D$ with $s \mapsto d$. Considering the composite $Z \to S \to D$ shows that $\mathrm{Reg}(D)$ is not empty and that the function $e \mapsto h^1(Y_e, T^1)$ is constant on D. Let $e \in \mathrm{Reg}(D)$. Then $\dim_e(D) \geq h^1(Y_e, T^1)$, since g is versal everywhere, and

$$\dim_e(D) \leq \dim_d(D) \leq h^1(Y_d, T^1) = h^1(Y_e, T^1).$$

So equality holds throughout, and we are done.

As for the last part we begin by noting that $s \mapsto b_n(X_s)$ is locally constant by smooth and proper base change and hence constant by the connectedness of S. Hence by the first part $\overline{S}$ is everywhere smooth. Assume that we can show that for every $s \in \overline{S}$ a deformation hull fulfills is tangent smooth (over $\mathbf{W}$). Then by Corollary 2.8 the closure of $S \bigotimes_{\mathbf{W}} \mathbf{W}[1/p]$ intersected with $\overline{S}$ is open and as it is obviously is closed and non-empty by assumption it equals S.

By Proposition 3.1 a deformation hull R is tangent smooth precisely when the deformation functor fulfills the TLP. To prove it, it suffices, as in the proof of Proposition 4.2, to show that $H^n_{DR}(\mathcal{X}/T)$ is free of rank $b_n(X_s)$ for all deformations $\mathcal{X}/T$ of X_s, $s \in S$. For this we may work locally and assume that $\overline{S}$ is the reduction of a smooth $\mathbf{W}$-scheme S' and we may also assume that we have a $\mathbf{W}$-homomorphism $R' := \widehat{\mathcal{O}}_{S',s} \to R$. By [B-O78, Prop. 3.15] there is a (unique) divided power structure on (R', pR') compatible with the standard structure on $\mathbf{W}$ (for the structure map $\mathbf{W} \to R$). The same thing is true for R and the map $R' \to R$ is a divided power map. Let us first consider the crystalline cohomology $R\Gamma(X/S')$. It is a perfect complex of $\mathcal{O}_{S'}$-modules (cf. [Be74, Thm. VII:1.1.1]). For each geometric point $\bar{t} \in S$ we then have that $R^n\Gamma(X/S') \bigotimes^L \mathbf{W}(\mathbf{k}(\bar{t})) = H^n(X_{\bar{t}/\mathbf{W}})$ (cf. [Be74, Cor. V:3.5.7]) and as that by assumption is free of rank b_n we get by Grauert's theorem $R^n\Gamma(X/S')$ is a free $\mathcal{O}_{S'}$ and commutes with base change. For any deformation $\mathcal{X} \to T$ we have a map $T \to \mathbf{Specf}\, R$ such that $\mathcal{X}$ is the pull back of a versal family and then $H^n(\mathcal{X}/T) = R^n\Gamma(X/S') \bigotimes \mathcal{O}_T$ again by [Be74, Cor. V:3.5.7]. □

PROOF of Theorem D: The characteristic p part follows directly from the proposition while the mixed characteristic part follows from it once one has notice that for points in the closure of the characteristic 0 locus we have by definition a torsion free lifting. □

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